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THE WAVEGUIDE EFFECT

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The main purpose of scattering theory is the study of qualitative features of scattered waves. In the present study we investigate anomalous effects of the type of the waveguide effect for scattering problems by one-dimensional periodic structures. According to the definition of R. M. Garipov, the waveguide effect consists of the existence of eigenwaves localized in the vicinity of the structure. The properties of these waves are described by generalized eigenfunctions, being solutions of problems for steady-state oscillations. We consider existence conditions and the possibility of a waveguide effect for one-dimensional periodic structures: for long waves on shallow water – a one-dimensional periodic underwater ridge of the plateau type; and for acoustic or electromagnetic waves – a one-dimensional periodic lattice of plates or smooth obstacles.*

1. Formulation of the Problem. Required Information. Let Γ describe on the plane $\mathbb{R}^2$ of Cartesian variables (x, y) the boundary between free space and an obstacle. It is assumed that Γ can be connected by a curve or a set of quite smooth closed or disconnected curves. It is assumed that Γ is periodic along the y axis with period 2π. The obstacle can be penetrable or impenetrable (Fig. 1).

Wave effects near the obstacle are described by a quite smooth complex function u(x, y) outside the obstacle boundary Γ, whose physical content is specific to the problem. Let Ω_1 and Ω_2 be the regions into which Γ divides the plane $\mathbb{R}^2$. The contraction of the function u(x, y) to regions Ω_1 and Ω_2 is denoted by $u_1(x, y)$ and $u_2(x, y)$, respectively. The functions $u_1(x, y)$ and $u_2(x, y)$ must be solutions of the Helmholtz equation:

$$(\Delta + \varkappa^2\lambda^2)u_1 = 0 \text{ in } \Omega_1,\ (\Delta + \lambda^2)u_2 = f \text{ in } \Omega_2. \tag{1.1}$$

The following matching conditions are satisfied on the boundary Γ of the regions Ω_1 and Ω_2:

$$\delta u_1 = u_2,\ \gamma\partial u_1/\partial \mathbf{n} = \partial u_2/\partial \mathbf{n} \text{ on } \Gamma. \tag{1.2}$$

Here $\varkappa > 0$, $\delta > 0$, $\gamma > 0$ are real, and λ is a complex parameter, whose physical meaning is determined by the content of the effect investigated. The function f(x, y) describes sources of oscillation, and is assumed periodic in y with period 2π and localized in the vicinity of the structure. All functions satisfy the condition of local finite energy [$u_1 \in W_{2\,loc}^1(\Omega_1)$, $u_2 \in W_{2\,loc}^1(\Omega_2)$], and are assumed periodic along the y axis with period 2π.

The general solution of the homogeneous Helmholtz equation with parameter λ, satisfying the periodicity condition along y with period 2π, is

$$u(x, y) = \sum_{k=-\infty}^{+\infty} [a_k^{\pm} \exp(iky + i|x|\sigma_k) + b_k^{\pm} \exp(iky - i|x|\sigma_k)], \tag{1.3}$$

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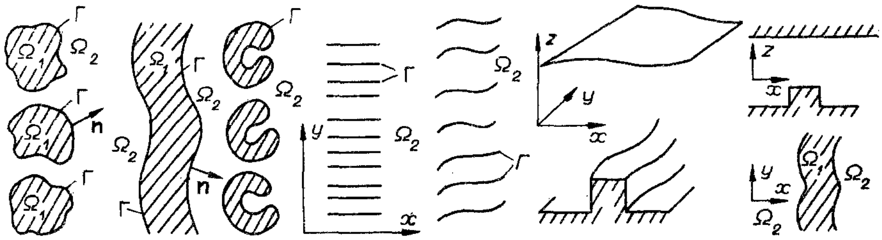

Fig. 1

where $\sigma_k = \sqrt{\lambda^2 - k^2}$; k is an arbitrary integer, and $a_k^{\pm}$ and $b_k^{\pm}$ are complex numbers.

For each fixed number k the expressions $\exp(iyk + ix\sigma_k)$ and $\exp(iyk - ix\sigma_k)$ are waves, not proceeding in one direction with respect to the x axis (the degenerate case $\sigma_k = 0$ or $\lambda^2 = k^2$ is possible). Therefore, if the function u(x, y) describes wave scattering at the obstacle Γ, it can be assumed that at sufficiently large distances from the obstacle the coefficients satisfy $b_k^+ = b_k^- = 0$ for all k (or $a_k^+ = a_k^- = 0$ for all k).

We further use

Definition 1.1. Condition (1.3), representing the solution of the Helmholtz equation with parameter λ, satisfying the conditions of 2π-periodicity in y, in which all coefficients b_k^+, b_k^- vanish ($b_k^+ = b_k^- = 0$), is called the radiation condition. A function of shape (1.3) satisfies the radiation condition of $b_k^+ = b_k^- = 0$ for all k. In relation (1.3) we select for $x \gg 1$ the number a_k^+, and for $x \ll -1$, the number a_k^-.

The terms describing the direction of wave propagation have a conditional nature, since time is reversible in the problems investigated in the present study. Considering problems describing scattering of plane waves by periodic structures, the coefficients b_k^+ and b_k^- in relation (1.3) must be given, since they describe incoming (not outgoing) waves.

Usually the sign in front of $\sqrt{\lambda^2 - k^2} = \sigma_k$ is selected in such a manner that the function u(x, y, λ) decreases upon moving away from the obstacle. The method of investigating the scattering problem, applied in the present study, is based on the idea of analytic continuation of the radiation condition, of fundamental solution and solvent in the parameter λ on the Riemann surface of analytic continuation. It was first used in [1] to investigate scattering problems by finite smooth obstacles.

The Riemann surface Λ of analytic continuation in λ of the function of shape (1.3) ($b_k^+ = b_k^- = 0$) was described in [2-4] and possesses the following properties: Λ has an infinite number of branches; the points λ = k (k = ±1, ±2, ...) are second-order branching points; for each element λ (λ ∈ Λ) there exists a number k_0, such that for all integers k ($|k| > k_0$) the inequality $\operatorname{Im}\sqrt{\lambda^2 - k^2} \geq 0$ is satisfied. We note that if the solution of the Helmholtz equation satisfies the radiation conditions, then, for some elements λ (λ ∈ Λ) it can increase with moving away from the obstacle.

Depending on the physical content of the scattering problem conditions (1.2) on the obstacle boundary Γ can be replaced by the Dirichlet condition (δ = 0)

$$u|_\Gamma = 0 \tag{1.2a}$$

or by the Neumann condition (γ = 0)

$$\partial u/\partial \mathbf{n}|_\Gamma = 0. \tag{1.2b}$$

For the upcoming discussion we need the following terminology [4]:

Definition 1.2. A quasi-eigenvalue of the scattering problem (1.1) [(1.2), or (1.2a), or (1.2b)], (1.3) is an element λ_* of the Riemann surface Λ, for which there exists a nontrivial solution of the corresponding homogeneous (f≡0) boundary-value problem.

We then have [2-6]

THEOREM 1.1. The set Λ_* of quasi-eigenvalues λ_* ($\lambda_* \in \Lambda_*$) of problem (1.1) [(1.2), or (1.2a), or (1.2b)], (1.3) is discrete on the Riemann surface Λ. Each quasi-eigenvalue has a finite multiplicity. For these λ values which are not quasi-eigenvalues ($\lambda \in \Lambda \backslash \Lambda_*$), the corresponding boundary-value problems have a unique solution.

The radiation conditions determine modes not proceeding in various directions along the x axis, including the case and only the case when for all k for which $a_k^+ \neq 0$ or $a_k^- \neq 0$ in (1.3), the following inequality is satisfied:

$$(\mathrm{Re}\,\lambda)(\mathrm{Re}\sqrt{\lambda^2 - k^2}) \geqslant 0. \quad (1.4)$$

For all k, sign $(\mathrm{Re}\sqrt{\lambda^2 - k^2})$ determines the propagation direction of oscillation modes, while sign $(\mathrm{Im}\sqrt{\lambda^2 - k^2}$ determines the damping or amplification of this mode upon moving away from the boundary. Conditions (1.4) make it possible to classify the quasi-eigenvalues according to the physical content of their corresponding quasi-eigenfunctions. We use the following terminology [4, 5]:

Definition 1.3. The eigenvalue of problem (1.1), (1.2), (1.3) or (1.1), [(1.2a) or (1.2b)], (1.3) is the quasi-eigenvalue λ_* $(\lambda_* \in \Lambda_*)$ for which conditions (1.4) are satisfied. The pseudo-eigenvalues are quasi-eigenvalues which are not eigenvalues. One can briefly say: quasi-eigenvalues = eigenvalues + pseudo-eigenvalues.

We have the following very important

Statement 1.1. The eigenvalues of the scattering problem by a one-dimensional periodic structure with conditions (1.1), (1.2a), (1.3) or (1.1), (1.2b), (1.3) can only be real numbers. For penetrable obstacles the proof of Theorem 2.1 of [4] does not apply, since in this case the solution can have discontinuities or lose smoothness on the boundaries; therefore, the conditions of the Holmgren theorem are not satisfied, and we have

Statement 1.2. The eigenvalue of the scattering problem is real when an infinite number of terms is contained in the radiation conditions (1.3) for the corresponding quasi-eigenfunctions.

A simple example is given in [5] of the existence of complex eigenvalues of scattering problems by a penetrable structure.

Statements 1.1 and 1.2 make it possible to formulate the waveguide effect for a one-dimensional periodic structure as the existence of eigenvalues and eigenfunctions of the corresponding scattering problems. The existence of an eigenfunction localized near the structure leads to a waveguide effect of the structure, and is possible only for a real eigen value. Therefore, the following discussion is devoted to the study of eigenvalues and eigenfunctions of scattering problems.

Comment 1.1. The association of the element λ_* $(\lambda_* \in \Lambda)$ to one branch or another of the Riemann surface Λ is intimately related to the shape of the quasi-eigenfunction corresponding to λ_*.

2. Periodic Chain of Resonators. The Waveguide Effect. Consider a possible waveguide effect for periodic structures of scattering problems, on which are described (1.1), (1.3) with conditions (1.2a) or (1.2b) on the boundary Γ. The possibility of a waveguide effect for one-dimensional structures is significant for various applied problems. The waveguide effect needs to be taken into account in investigating water waves near a periodic coastline, as well as in investigating propagation of acoustic or electromagnetic waves near a periodic lattice or a periodic surface.

Let Γ be a one-dimensional periodic chain of resonators with period 2π [Fig. 2a, b: there is no resonator interior, and there exists only a channel (aperture); c, d: cavity of resonator Ω_{int}]. The exterior is denoted in all cases by Ω_{ext}, the channel (aperture) by Ω_ε, and the characteristic length ℓ, width ε. The ε value can be quite small in comparison with ℓ and other characteristic resonator sizes. For the following discussion the following theorem is needed, whose proof is contained in [4]:

THEOREM 2.1. If ν^2 is an eigenvalue of the Laplace operator Δ in the region Ω_{int} (the interior of a resonator) for functions satisfying the Dirichlet or Neumann conditions on the resonator walls $(\partial\Omega_{int})$, then there exists a small quasi-eigenvalue $\lambda^*(\varepsilon)$ corresponding to the scattering problem (1.1), [(1.2a) or (1.2b)], (1.3), so that we have

$$\nu = \lim_{\varepsilon \to 0} \lambda^*(\varepsilon). \quad (2.1)$$

Besides, in the Neumann problem (1.1), (1.2b), (1.3) there exist for all k quasi-eigenvalues such that

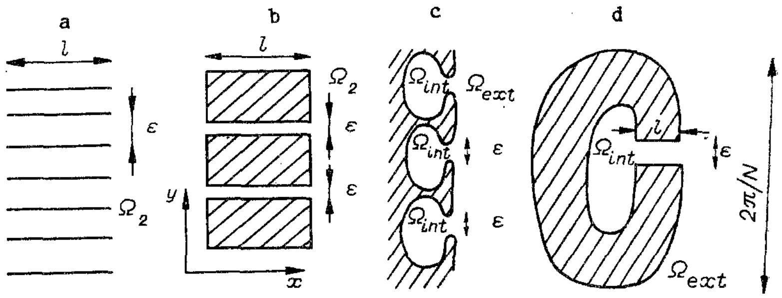

Fig. 2

$$\lim_{\varepsilon\to 0} \lambda_k^*(\varepsilon) = k\pi/l. \tag{2.2}$$

According to (2.1) and (2.2) the limits are implied in the topology of the Riemann surface Λ. The numbers $k\pi/\ell$ in relation (2.2) are related to the eigen-oscillations of an open channel. If the wave propagation velocity is equal to 1, then $k\pi/\ell$ are the eigenfrequencies of the channel for quite small ε. Theorem 2.1 discusses the existence of quasi-eigenvalues on the Riemann surface Λ and the nearness to the real axis in some special cases.

For convenience of the following discussion we write the formulation of the Neumann (or Dirichlet) problem, describing the quasi-eigenvalues λ^* and the quasi-eigenfunctions $u^*(x, y)$ of the scattering problem (1.1), (1.2b), (1.3):

$$\begin{gathered}(\Delta + \lambda^2)u = 0 \text{ in } \Omega = \Omega_{\text{int}} + \Omega_{\text{ext}} + \Omega_\varepsilon, \\ u(x, y + 2\pi) = u(x, y),\ \partial u/\partial \mathbf{n} = 0 \text{ (or } u = 0) \text{ on } \Gamma, \\ u(x, y) = \sum_{k=-\infty}^{+\infty} a_k^\pm \exp\left(iyk + i|x|\sqrt{\lambda^2 - k^2}\right),\ |x| \gg 1.\end{gathered} \tag{2.3}$$

A. Scattering problems by a one-dimensional periodic lattice of plates have important applications in the areas of aeroacoustics [6-9] and electrodynamics [10].

Let Γ be a lattice of segments, parallel to the x axis, periodic in the direction of the y axis with period 2π, modeling a one-dimensional periodic lattice of plates. The functions $\exp(i\lambda x)$ and $\exp(-i\lambda x)$ satisfy homogeneous Neumann conditions on Γ and are solutions of the homogeneous Helmholtz equation with parameter λ. Let Λ_0 be the branch of the Riemann surface Λ with cuts $(-\infty, -1)$ and $(1, +\infty)$, on which the inequality $\operatorname{Im}\sqrt{\lambda^2 - k^2} > 0$ is satisfied for all k. Due to Theorem 2.1, for certain values of ℓ (the plate length) and ε (the separation between any two plates) (Fig. 2a) there exist on the branch Λ_0 quasi-eigenvalues λ_* of problem (2.1). Let $u_*(x, y, \lambda_*)$ be the quasi-eigenfunction corresponding to λ_*, $\operatorname{Im}\sqrt{\lambda_*^2 - k^2} \neq 0$. Since $u_*(x, y)$ is the solution of (2.1), for sufficiently large $|x|$ ($|x| \gg 1$) we have

$$u_*(x, y) = \sum_{k=-\infty, k\neq 0}^{+\infty} a_k^\pm \exp\left(iky + i|x|\sqrt{\lambda_*^2 - k^2}\right) + a_0^\pm \exp(i|x|\lambda_*).$$

Therefore, the function $v_* = u_* - a_0^+\exp(ix\lambda_*) - a_0^-\exp(-ix\lambda_*)$ is the solution of problem (2.3) when Γ describes a lattice of plates. We note that if the function $u_*(x, y, \lambda_*)$ increases upon moving away from the obstacle, then $v_*(x, y, \lambda_*)$ decreases. Therefore, it can be assumed that the quasi-eigenfunctions of the scattering problem by a lattice of plates do not increase with moving away from the lattice of plates if λ is a real number, and decrease if $\operatorname{Im}\lambda \neq 0$. By means of the Green's formula one obtains a relation for the quasi-eigenfunctions $u_*(x, y, \lambda_*)$, $\lambda_* \in \Lambda_0$:

$$\int_{\Omega_0} \left(|\nabla u_*|^2 - \lambda_*^2 |u_*|^2\right) d\Omega_0 = 0. \tag{2.4}$$

Here $\Omega_0 = \Omega \cap \{(x, y), 0 \le y < 2\pi\}$. From (2.4) follows the equality $\operatorname{Im}\lambda_* = 0$. The existence of real quasi-eigenvalues λ_* on the branch Λ_0 follows from (2.2) for sufficiently small ε if $|k\pi| < \ell$ $(k = \pm 1, \pm 2, \ldots, \pm k_0)$.

Let k_0 be the maximum integer for which the inequality $|k_0\pi| < \ell$ is satisfied, and let ℓ be the number of segments modeling the lattice of plates (see Fig. 2). We then have

THEOREM 2.2. In the vicinity of each number ν_k, $\nu_k = k\pi/\ell$ ($k = \pm 1, \pm 2, \ldots, \pm k_0$) there exists for sufficiently small ε ($\varepsilon > 0$) a real eigenvalue λ_k^* of problem (2.3), corresponding to an eigenfunction $u_k^*(x, y, \lambda_k^*)$ localized in the vicinity of the lattice Γ, in the sense that $u_k^*(x, y, \lambda_k^*) \to 0$ for $|x| \to \infty$.

The presence of real eigenvalues of problem (2.3) for a lattice of plates guarantees the possibility of a waveguide effect for the corresponding structures. Therefore, the statement of Theorem 2.2 describes the possibility of existence of a waveguide effect for a one-dimensional periodic lattice of plates, on which the homogeneous Neumann condition is satisfied.

B. Let the obstacle Γ possess the property

$$\Gamma = \Gamma + (2\pi/N)\mathbf{e}_y. \tag{2.5}$$

Here and in the following N is a natural number, and e_y is the unit vector in the direction of the y axis. Relation (2.5) implies that the structure describing Γ has a period along the y axis, equal to $2\pi/N$. In Fig. 2, N is the number of identical resonators in the band $y_0 \le y \le y_0 + 2\pi$. By a direct calculation it can be verified that if $u(x, y, \lambda^*)$ is a quasi-eigenfunction of (2.3), then $u(x, y + 2\pi/N, \lambda^*)$ is also a quasi-eigenfunction of problem (2.3), when the one-dimensional periodic structure Γ is of order $2\pi/N$ (2.5). The whole solution $u(x, y)$ of problem (2.3) when Γ satisfies condition (2.5) can be represented in the form

$$u(x, y) = \sum_{n=1}^{N} u_n(x, y).$$

The functions $u_n(x, y)$ satisfy the quasiperiodicity conditions ($n = 1, \ldots, N$)

$$u_n(x, y + 2\pi/N) = u_n(x, y)\exp(i2\pi n/N). \tag{2.6}$$

These relations describe a specific oscillation near the one-dimensional periodic structure, whose repetition period is smaller than the period along the structure axis of the unknown solution. One usually says that (2.6) describes an oscillation phase shift in adjacent regions near the one-dimensional periodic structure.

Let $u_n(x, y, \lambda)$ be the solution of problem (2.3), satisfying relation (2.6), $n \in \{1, 2, \ldots, N\}$. Then the following representation is valid sufficiently far from the structure ($|x| \gg 1$)

$$u_n(x, y, \lambda) = \sum_{k=-\infty}^{+\infty} c_k^{\pm} \exp\left[i(n + kN)y + i|x|\sqrt{\lambda^2 - (n + kN)^2}\right]. \tag{2.7}$$

The coefficients c_k^+ and c_k^- in expansion (2.7) are related to the coefficients a_k^+ and a_k^- in the radiation conditions (2.3) by the relations $c_k^+ = a_{n+kN}^+$, $c_k^- = a_{n+kN}^-$. The functions of shape (2.7) depend analytically on the parameter λ on the Riemann surface. The numbers $\pm(n + kN)$ are the branching points. A substantial deviation of (2.7) from the radiation condition (2.3) is the absence for $n \ne N$ of a "nonvanishing" term of the form $a_0^{\pm}\exp(i|x|\lambda)$. The statement in the following lemma follows from the condition of selecting the branch Λ_0 and from (2.7).

Lemma 2.1. If the quasi-eigenfunction $u^*(x, y, \lambda^*)$ satisfies for some number n ($n \in \{1, 2, \ldots, N - 1\}$) condition (2.7), then $u^*(x, y, \lambda^*)$ decreases with moving away from the obstacle ($|x| \gg 1$) for all quasi-eigenvalues λ^*, located on the branch Λ_0 of the Riemann surface Λ.

The following theorem is valid for Fig. 2b:

THEOREM 2.3. If m is the largest natural number for which the inequality $m\pi/\ell < \min\{n, N - n\}$ is satisfied, then for sufficiently small ε there exist real eigenvalues (2.3) $\lambda_k^*(\varepsilon)$ ($-m \le k \le m$), such that the relation $k\pi/l = \lim_{\varepsilon\to 0}\lambda_k^*(\varepsilon)$ is satisfied. The eigenfunctions $u_k^*(x, y, \lambda_k^*$ are localized in the vicinity of the structure, and satisfy relationship (2.6).

Proof. The existence of quasi-eigenvalues follows from Theorem 2.1. It follows from Lemma 2.1 that these quasi-eigenvalues are eigenvalues, and the eigenfunctions are localized in the vicinity of the structure shown in Fig. 2b. If the structures of Fig. 2c, d satisfy condition (2.5), then the following is valid.

THEOREM 2.4. Let ν_k^2 be the eigenvalue of the Laplace operator Δ in the region Ω_{int} for functions satisfying the Dirichlet or Neumann conditions on the internal boundary $\partial\Omega_{int}$, and let m be the largest natural number for which the inequality $|\nu_m| < \min\{n, N - n\}$ is satisfied. It is assumed that ν_k (k = 1, 2, ...) are sorted in increasing order ($|\nu_1| \le |\nu_2| \le \ldots \le |\nu_k| \le |\nu_{k+1}| \le \ldots$). For a sufficiently small ε there exist real eigenvalues $\lambda_k^*(\varepsilon)$ of problem (2.3) ($-m \le k \le m$), such that the relations $\nu_k = \lim_{\varepsilon\to 0}\lambda_k^*(\varepsilon)$ are satisfied. When ν_k are quasi-eigenvalues of problem (2.3) in the region Ω_{ext}, the eigenfunctions corresponding to $\lambda_k^*(\varepsilon)$ are localized in the vicinity of the structure Γ.

The proof is similar to the preceding one, since Lemma 2.1 is valid for the structures of Fig. 2c d.

Comment 2.1. Since N (the number of identical resonance substructures in a period) can be arbitrarily large and the number n, determining the oscillation phase shift in neighboring substructures (2.6) is arbitrary (0 < n < N), then $\min\{n, N - n\}$ can be quite large.

3. Underwater One-Dimensional Periodic Structure. The Waveguide Effect. In 1957, M. A. Lavrent'ev offered the hypothesis that the nonuniformity of the type of homogeneous underwater coast lines can serve as waveguides for surface waves on water. In 1965, Garipov [11] showed, within the linear theory, that bottom surfaces, homogeneous in one of the variables, can indeed be waveguides for surface waves. More detailed results were discussed in [12]. This process was investigated experimentally in [13]. Various aspects of the scattering problem by a homogeneous coast line were investigated in [14-16].

In the present study we have considered, within the linear theory of the acoustic approximation [12], existence conditions and the possibility of a waveguide effect for underwater one-dimensional periodic structures. In this case, the propagation of surface waves in shallow water is described by the equation [12]

$$\frac{\partial^2\zeta}{\partial t^2} - g\left\{\frac{\partial}{\partial x}\left(h\frac{\partial\zeta}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial\zeta}{\partial y}\right)\right\} = f, \tag{3.1}$$

where ζ is the height of the fluid layer; g, the free-fall acceleraton; h, depth; f, the perturbation constant; and (x, y), Cartesian coordinates. Let Ω_1 be the region in the (x, y) plane, periodic along the y-axis and bounded along the x-axis. The region Ω_1 models the ridge projection on the (x, y) plane (Fig. 3). Let Ω_2 be the complement to the region Ω_1 in the (x, y) plane. It is assumed that the regions Ω_1 and Ω_2 are periodic along the y axis with period $2\pi/N$. If the depth is constant and equal to h_1 and h_2 in the regions Ω_1 and Ω_2, respectively, and the time dependence is of the form $\zeta(x, y, t) = \zeta(x, y)\exp(-i\omega t)$, then by (3.1) the function $\zeta(x, y)$ must satisfy the relations

$$(\Delta + \omega^2/gh_1)\zeta_1 = 0 \text{ in } \Omega_1,\ (\Delta + \omega^2/gh_2)\zeta_2 = f \text{ in } \Omega_2. \tag{3.2}$$

Here the subscripts 1 and 2 denote the constriction of the corresponding functions to the regions Ω_1 and Ω_2. Let Γ be the boundary of the regions Ω_1 and Ω_2. On the line Γ the depth varies jumpwise and, therefore, the following relations [17, 18] must be satisfied:

$$\zeta_1 = \zeta_2,\ h_1\partial\zeta_1/\partial\mathbf{n} = h_2\partial\zeta_2/\partial\mathbf{n} \text{ on } \Gamma \tag{3.3}$$

(n is the unit vector normal to the boundary Γ). It is assumed that the functions ζ_1 and ζ_2 satisfy the periodicity conditions along the y axis with period 2π, $\zeta(y + 2\pi) = \zeta(y)$. The function f is also assumed to be periodic along the y axis with period 2π. Besides, it is assumed that f is localized in the vicinity of the structure, while the function ζ satisfies the radiation conditions for large $|x|$ ($|x| \gg 1$). We have

$$\zeta = \sum_{k=-\infty}^{+\infty} a_k^{\pm}\exp\left(iky + i|x|\sqrt{\lambda^2 - k^2}\right), \tag{3.4}$$

where $\lambda = \omega/\sqrt{gh_2}$; a_k^+ and a_k^- are complex numbers; one chooses a_k^+ if $x \gg 1$ and a_k^- if $x \ll -1$.

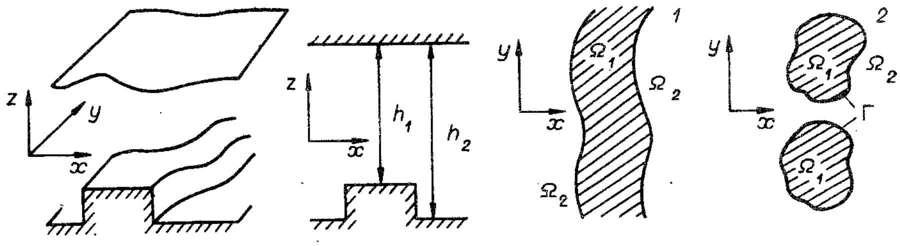

Fig. 3

Relations (3.2)-(3.4) are a special case of problem (1.1)-(1.3), but the asymptotic behavior of the solution of this problem has a special shape, depending on h_1 ($h_1 \to \infty$) or h_2 ($h_2 \to \infty$), since this parameter appears both in the equation and in the boundary condition. We have [5]

THEOREM 3.1. The set of quasi-eigenvalues of problem (3.2)-(3.4) is discrete on the Riemann surface Λ. If $h_2 \to \infty$, the quasieigenvalues of problem (3.2)-(3.4) (here and in the following $h_1 = 1$) are near the eigenvalues of the Dirichlet problem in the region Ω_1 in the class of functions satisfying the periodicity conditions. All convergences are implied on some compact Riemann surface Λ with induced topology of Λ.

This theorem makes it possible to describe qualitatively the behavior of the solution of the scattering problem on an underwater coast line when the frequency of the problem is close to that mentioned in the formulation of the theorem. The asymptotic localization of the quasi-eigenvalues and quasi-eigenfunctions in the vicinity of the corresponding eigenvalues and eigenfunctions can be understood as a waveguide effect for the structure investigated. It is advisable, however, to investigate the possibility and existence conditions of eigenvalues and eigenfunctions of problem (3.2)-(3.4), since they describe the waveguide effect in pure form.

Let Ω_2 be a connected region; then Ω_1 models a periodic chain of cavities if $h_1 > h_2$, or a chain of underwater plateaux if $h_2 > h_1$. We have

THEOREM 3.2. When the region Ω_1 models a chain of underwater mountains or cavities, problem (3.2)-(3.4) has only real eigenvalues.

Proof. By Contradiction. Two cases are possible: 1) under the radiation conditions (3.4) an infinite number of terms does not vanish, when the proof is similar to that of Theorem 2.1 of [4]; 2) under the radiation conditions (3.4) only a finite number of terms are nonvanishing. Let λ_* be a complex quasi-eigenvalue, and let $v_*(x, y, \lambda_*)$ be a quasi-eigenfunction of the problem. For $x \gg 1$ let there be

$$v_*(x, y, \lambda_*) = v_2(x, y, \lambda_*) = \sum_{n=-n_0}^{+n_0} a_n^+ \exp\left(iny + ix\sqrt{\lambda_*^2 - n^2}\right). \tag{3.5}$$

A function $v_2(x, y, \lambda_*)$ of form (3.5) can be defined for all numbers x, y, $(x, y) \in \mathbb{R}^2$, and $v_2(x, y, \lambda_*)$ is the solution of the Helmholtz equation for all x, y. Therefore, a function $w = v_* - v_2$ can be found, satisfying the Helmholtz equation for all x, y outside the boundary Γ [$(x, y) \notin \Gamma$], and identically vanishing for sufficiently large x ($x \gg 1$). By the Holmgren theorem it then follows that the contraction of the function $w(x, y, \lambda_*)$ to the region Ω_2 vanishes identically ($w|_{\Omega_2} \equiv 0$), while Ω_2 is a connected region, which contradicts the radiation conditions.

Let the region Ω_1 consist of N connected components in the band $0 \le y < 2\pi$ and, besides, it is assumed to be periodic along the y axis with period $2\pi/N$. Then the following is valid:

THEOREM 3.3. The waveguide effect may take place for a one-dimensional chain of mountains.

Proof. Let the region Ω_1 be periodic along the y axis with period $2\pi/N$ ($N > 1$). A solution of the scattering problem (3.2)-(3.4) when the sources satisfy condition (2.6)

$$f(x, y + 2\pi/N) = f(x, y) \exp(i2\pi n/N)$$

for a natural number n ($1 \le n < N$) can be sought in the class of functions for which (2.6) is valid.

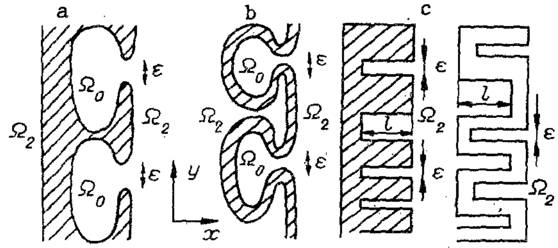

Fig. 4

For the quasi-eigenvalues of problem (3.2)-(3.4) in the class of functions with conditions (2.6) Theorems 3.1 and 3.2 remain valid. It is necessary to point out that the region Ω_1 is disconnected, and has no less than N connected components in the interval $0 < y < 2\pi$.

An underwater periodic mountainous coast line is described by the relation $h_2 \gg h_1$. Let $\mu_k{}^2$ be an eigenvalue of the Dirichlet problem in the region Ω_1; more accurately, in one of the connected components of the region Ω_1. For large h_2 we find quasi-eigenvalues $\lambda_k{}^*$, so that $|\mu_k - \lambda_k{}^*\sqrt{h_2}| < \sqrt{h_2}\min\{n, N - n\} - |\mu_k|$. In this case it is sufficient to investigate those μ_k lying in the interval $-\sqrt{h_2}\min\{n, N - n\} < \mu_k < \sqrt{h_2}\min\{n, N - n\}$ $(k = 1, 2, \ldots, k_0)$, while Lemma 2.1 holds for the corresponding quasi-eigenvalues $\lambda_k{}^*$. Hence follows what was to be proved.

Comment 3.1. The number N of identical resonance structures in the band $0 \leq y < 2\pi$ can be arbitrarily large, and the number n, determining the oscillation phase shift in neighboring substructures of a periodic obstacle Ω_1 is, generally speaking, arbitrary, whence $\min\{n, N - n\}$ can be arbitrarily large. Therefore, since the quasi-eigenvalues depend discontinuously on the parameters h_1 and h_2 if $h_1 \neq h_2$, $h_1 > 0$, $h_2 > 0$ (the requirements of [19] are satisfied), the conditions of Lemma 2.1 are satisfied for a sufficiently wide choice of possible h_1 and h_2.

For a connected region Ω_1 it can be shown that for underwater obstacles of special shape (Fig. 4) a waveguide effect takes place. Let a structure of an underwater obstacle have a shape determined by a structural size of period $2\pi/N$ and by an ε value characterizing the mountainous underwater resonator. We then have

THEOREM 3.4. For sufficiently large h_2 and sufficiently small ε, the structures of Fig. 4a-c possess a waveguide effect.

Proof. For sufficiently large h_2 (in this case Ω_1 models an underwater coastline) the quasi-eigenvalues of problem (3.2)-(3.4) are near the quasi-eigenvalues of the Neumann problem in the region Ω_2, which for sufficiently small ε are, in turn, close to the eigenvalues of the Neumann problem in Ω_0 and corresponding to the numbers describing the eigenoscillations in the mountainous resonator [4]. Therefore, the conditions of Lemma 2.1 are satisfied for sufficiently large h_2 and small ε for the quasi-eigenvalues. Consequently, in this case the structures of Fig. 4a-c possess a waveguide effect, which was required to be proved.

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DEVELOPMENT OF THERMOCAPILLARY CONVECTION IN A FLUID CYLINDER AND CYLINDRICAL AND PLANE LAYERS UNDER THE INFLUENCE OF INTERNAL HEAT SOURCES

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Under weightless conditions, neither external forces nor forces associated with self-gravitation are strong enough to cause convective motion. However, convection may develop due to the fact that surface tension is dependent on temperature.

The studies [1-4] investigated the conditions for the development of convection in a fluid during the heating of a solid or free surface. Here, we study the stability of the equilibrium state which develops in a liquid cylinder and cylindrical and plane layers under the influence of constant internal heat sources. Explicit formulas are obtained for the critical Marangoni numbers. It is shown that allowance for deformation of the free surface leads to a decrease in stability and the appearance of a discontinuity on the neutral curve. Also, the equilibrium state of the plane layer is more stable than in the analogous Pearson problem [1].

1. Fluid Cylinder. Let a quiescent fluid cylinder contain constant internal heat sources of intensity q. Then the equilibrium state is described by the formulas

$$u = v = w = 0,\ p = \text{const},\ \Theta(r) = -qr^2/(4\chi) + \text{const}. \quad (1.1)$$

Here, (u, v, w) are components of the velocity vector in the cylindrical coordinate system (r, φ, z); p is pressure; Θ is temperature; χ = const is the diffusivity of the fluid.

As the characteristic scales of length, velocity, pressure, and temperature, we choose the quantities b, ν/b, $\rho\nu^2/b^2$, and $\nu\gamma b/\chi$ (b is the radius of the cylinder, ν is kinematic

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